所別：水文科學研究所碩士班 一般生 科目：應用數學

1. Find a unit vector $\hat{n}$ normal to each of the following surfaces:

   (a) [5%] $z = 2 - 2x - y$ (b) [5%] $z = \left(1 - 3x^2\right)^{1/2}$

2. Calculate the divergence of each of the following vector functions $\mathbf{F}$:

   (a) [5%] $\mathbf{F} = \mathbf{i}2yz + \mathbf{j}3xz + \mathbf{k}xy$

   (b) [5%] $\mathbf{F} = \left(-\mathbf{i}3xy + \mathbf{j}x^2\right)/\left(x^2 + y^2\right),$ $(x, y) \neq (0, 0)$

3. [20%] Let $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ be unit vectors in Cartesian coordinates and

   $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta,$ and $\hat{\mathbf{e}}_z$ be unit vectors in cylindrical coordinates. Show that

   $$\mathbf{i} = \hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta,$$
   $$\mathbf{j} = \hat{\mathbf{e}}_r \sin\theta + \hat{\mathbf{e}}_\theta \cos\theta,$$
   $$\mathbf{k} = \hat{\mathbf{e}}_z$$

4. Solve the following ordinary differential equations for $y = y(x)$:

   (a) [10%] $\cos y \dfrac{dy}{dx} = \dfrac{-x\sin y}{1 + x^2},$ and $y(1) = \dfrac{\pi}{2}.$

   (b) [10%] $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$, and $y(0) = 0,$ $\dfrac{dy(0)}{dx} = 0.$

   (c) [20%] $\dfrac{d^2 y}{dx^2} - \dfrac{2x}{1 + x^2}\dfrac{dy}{dx} + \dfrac{2}{1 + x^2}y = 1 + x^2$

5. [20%] Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$